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The Social Construction of Mathematical Continuity: A Socioepistemological Approach

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Abstract

In this paper we describe the empirical results of a qualitative study conducted within the theoretical framework of Socioepistemology Theory for Mathematics Education (Cantoral, 2013) in which we analyzed the role played by gesture in the processes of social construction of mathematical knowledge. It was assumed that special analysis of discourse is a mean to produce knowledge on the possible changes in the pedagogical treatment of the mathematical notion of point-continuity. The implementation of the design was carried out with eight university students between 19 and 21 years of age. The basic requirement was to have certain knowledge of the elementary functions of calculus as well as previous contact with the mathematical concept of continuity. We identify some elements that were essential in the formulation of the students' responses, at individual as well as group level; for example, the incidence of the gesture aspect is a means that offers the possibility to articulate the visualization of mathematical concepts, so that a representation on the computer screen only establishes the scenario where the student will expand and generate new meanings even more, if these representations are articulated with gestures.

Keywords: Discourse /gesture, mathematical continuity of a function, Socioepistemology, Social practice.

1. Introduction

In recent decades, the interest in research into phenomena related to the learning of calculus (differential and integral) has increased. Artigue (1995) points out the complexity of this problem referring to its beginnings during the twentieth century motivated by the massive incursion of the teaching of calculus in French schools. Hoffman, Johnson and Logg (2004), discuss a possible crisis in the contemporary teaching of calculus in universities and the need for a paradigm shift. This change is the result of new information technologies especially considering the benefits provided by computational mathematics.

Various studies focused on analyzing learning math from a social perspective of knowledge (Rasmussen and Kwon, Rasmussen, Stephan, and Allen, 2004; Cantoral and Farfán, 2004; Cantoral, 2013; Cordero, Silva-Crocci, 2012). For example, Sierpinska and Lerman (1996) points out that math is a social creation and it is therefore necessary to understand what kind of social relations must undergo and how students should organize the classes they appropriate or construct knowledge. Similarly, our research group has approached the problem and considered *mathematical scholar discourse* to be of unique importance (Cordero, 2005, Cordero, Cen y Suárez, 2010). Discourse analysis provides a means to generate understanding about the

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nature of the phenomena associated with teaching advanced mathematics and provides basis for proposals of the redesigning school knowledge.

It is assumed as a hypothesis that in current mathematical scholar discourse, in science and engineering programs, more time is dedicated to the teaching of function, derivative and integral than to other mathematical concepts. Such concepts are interconnected thanks to the Cauchy proposal of limit, which not only comprises the core of calculus but even more complex notions that involve infinite processes. For example, in the last years of college the notion of punctual continuity is dealt with as a rigorously algorithmic technique only applied in continuous functions; the notion is therefore reduced to an algebraic treatment, direct substitution: $\mathbf{lim}_{x \to a} f(x) = f(a)$**.** For example, for the function $f(x) = 3x^2 - x + 2,$ **,** when x tends to 1, the result is the direct substitution $\mathbf{lim}_{x \to 1} f(x) = f(1) = 3(1^2) - (1) + 2 = 4$.

However, understanding and using algebra is dependent on understanding a number of fundamental concepts, one of which is the concept of equality (Knuth, 2006). In addition, at university level, a definition of punctual continuity is used based on a particular logic of quantifiers and a new interpretation of the concept of equality. It has been detected that students face their first difficulties with understanding the notion of limit, when studying the concept of punctual continuity «continuous functions at a point». Several studies report that the difficulties are accompanied by aspects of diverse nature, whether epistemological, cognitive or didactic (Aparicio and Cantoral, 2006; Artigue, 1995; Bezuidenhout, 2001; Crespo, 2004; Azcárate and Delgado, 1996; Farfán, 2012; Hitt, 1994; Raman, 2004; Sierra, M., González, M., López, C, 2000; Tall and Vinner, 1981).

Aparicio and Cantoral (2006) in particular stressed that the concept of punctual continuity and the structure followed in its teaching, seems not to form an appropriate basis for the construction of meaning associated to global continuity, because the “strange” notion of a continuous function at a point, is contradictory to the aprioristic nature of global continuity in the spontaneous reasoning of the students. In other words, it goes against the way humans perceive the nature of physical change in the study of real phenomena. Such changes, we assume, are perceived in global terms and are not focused on locally. Therefore, we share Vergnaud (1990)’s view, when speaking about basic education, and which we have extended to higher education, that a mathematical concept to be learned should not be reduced to a simple definition.

The purpose of this study was to analyze the gestures to provide empirical evidence of the convenience of incorporating cultural aspects in the treatment of the continuity school. Thus, we extend the study by Aparicio and Cantoral (2006) in which it is reported that the analysis of socio-cultural aspects such as gesture and speech provides information on socialization processes of mathematical meanings.

2. Theoretical Orientation

This paper describe the possibility of re-designing scholar mathematical discourse of the notion of punctual continuity in calculus textbooks and obviously, on all educational practices, starting from the consideration of elements originating from social practice linked to the notion of punctual continuity. To do so, we lean towards socioepistemology for orientation, which is a theoretical approach that seeks to explain the didactic phenomena taking place in the field of mathematics by analyzing the role played by the social construction of knowledge under a systemic approach that specifies the incorporation of aspects like communication, the pursuit of consent, the construction of languages or the design of tools for the study of this phenomena (Cantoral, 2013).

Let us say that the notion of punctual continuity has an eminently aprioristic nature in the spontaneous reasoning of human beings. In everyday social terms continuity, admits the connotation of quality, that is to say, it is understood as the state of permanency or the non-interruption of a certain process and not as their study or local property. In this sense, an experience such as the free movement of the hand from one side to another is conceived as the continuous descriptive trajectory of its motion. The hand then, travels all the intermediate points between one end and the other along its trajectory. In the same way, free-falling solids are thought to go through all the intermediate points of their trajectory.

The current school mathematics discourse, does not consider the wealth of such experiences in the treatment of punctual continuity. As a rule, it leans more towards developing certain types of algorithms and techniques and infusing the use of an excessive formal language associated with the idea of mathematical rigor and students’ ability to abstract. In our opinion, this causes the student to conceive the meaning, necessity, use and intention of such a concept, erroneously. Undoubtedly, a belief persists that the development of algorithmic abilities for the resolution of certain types of mathematical problems must be preserved and any other types of elementary experiences in the learning process are left aside (Cantoral and Farfán, 2003, 2004).

The kind of thinking and school practices set around the notion of punctual continuity should correspond to thoughts more of the space-relational type, and to the associated discursive forms such as gesticulation. Also, we consider that in-depth study of non-oral communication codes, such as gesticulation, in the learning of mathematics gives relevancy and intentionality to the school context.

In Cazden (1986) language is recognized as the means that relates the cognitive to the social in such a way that cognitive and linguistic developments are considered to be socially conditioned (Green, 1998; Hicks, 1995). On the other hand, in Kress and Ogborn (1998) language is analyzed as a form that, in interaction with other modal forms, enables us to see the representation of knowledge and mental states and communication in educational contexts. They suggest that the role played by verbal, written, or figural or attitudinal discourse is cultural. The work developed in discursive psychology refers to speech as an action located in a discursive context that builds meaning, reality and even, cognition itself (Candela, 2001). Our theoretical approach recognizes gesture as part of discourse and we therefore consider it in the study of the construction of mathematical knowledge, (Cantoral, Farfán, 2003; Cantoral, 2013).

The mathematical continuity concept

Below we show how punctual continuity is presented in two principal textbooks of calculus and two textbooks of analysis in Mexican Educational System:

« Stewart, J. (2001). Calculus. Early Transcendentals. 4th Edition. p. 122 »

"We will see that the mathematical definition of continuity intimately corresponds to the meaning of the word continuity in the daily language. (A continuous process that takes place gradually, without interruption or steep change) […] "Quotations from the author

Definition: A function f is continuous in a number a if $\lim_{x \to a} f(x) = f(a)$

"If f is not continuous in a, we say that f is discontinuous in a or that f has a discontinuity in a […] " quotations from author

« Spivak, M. (1998). Calculus. Cálculo Infinitesimal. Reverté. 2nd edition. p. 142 »

Definition: Function f is continuous in a if $\lim_{x \to a} f(x) = f(a)$.

« Ross, K. (1980). Elementary Analysis: The Theory of Calculus. Springer-Verlag. p. 88 »

Definition: Let f be a real-valued function whose domain is a subset of $\Re$. The function f is continuous at x_0 in *dom*(f) if, for every sequence (x_n) in *dom*(f) converging to x_0, we have $\lim f(x_n)=f(x_0)$. If f is continuous at each point of a set $S \subseteq$ *dom*(f), then f is said to be continuous on S. The function f is said to be continuous if it is continuous on *dom*(f).

Theorem: Let f be a real-valued function whose domain is a subset of $\Re$. Then f is continuous at $x_0 \in$ *dom*(f) if and only if for each ε>0 there exists δ>0 such that

$$x \in dom(f) \text{ and } |x - x_0| < \delta \text{ imply } |f(x) - f(x_0)| < \varepsilon$$

« Royden, H. (1988). Real Analysis. Macmillan Publishing Company, New York. Pp. 47 »

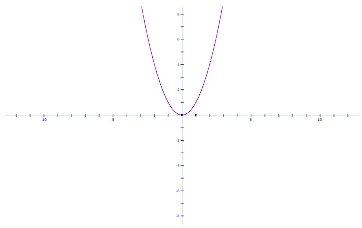
Let f be a real-valued function whose domain of definition is a set E of real numbers. We say that f is continuous at the point x in E if given ε>0, there is δ>0 such that for all y in E with $|x - y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. The function f is said to be continuous on a subset A of E if it is continuous at each point of A. If we merely say that f is continuous, we mean that f is continuous on its domain.

Notice that the first two books coincide in how they present or define the concept of continuous function at a point preferring to keep it brief and simple. The two remaining books opt to present a more detailed definition with a certain degree of mathematical formality. In both cases, global continuity (continuity on an interval) is presented as an immediate consequence of punctual continuity or an extension of this property. In this sense, the concept of "punctual discontinuity" will not be defined but characterized.

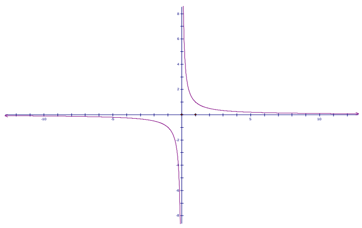
In a same way, is present the types of difficulties that students have when dealing with the concept of continuous function at a point. We will do this by considering the findings reported in two studies. First, let us consider the work developed by Tall and Vinner (1981) who differentiate between the formal definition of a concept and its perception by the student. This work mentions that students rarely refer to the concept of continuity based on its formal definition, but that the concept image is built starting from the informal use of such a term. This way, it is assumed that the student's notion of continuous function is global, on an interval and not at a point. Their report discusses, in fact, the uses of continuity. Figure 1 shows the questionnaire that the authors applied to 41 students which asked: "Which of the following functions are continuous? If possible, give your reason for your answer".

Figure 1

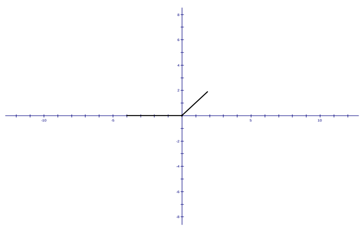
$$f_1(x) = x^2$$



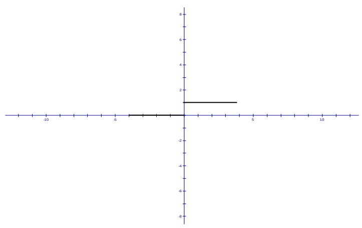
$$f_2(x) = 1/x \ \ (x \neq 0)$$



$$f_3(x) = \begin{cases} 0 (x \leq 0) \\ x (x \geq 0) \end{cases}$$



$$f_4(x) = \begin{cases} 0 (x \leq 0) \\ 1 (x > 0) \end{cases}$$



$$f_5(x) = \begin{cases} 0 \text{ (}x\text{ rational)} \\ 1 \text{ (}x\text{ irational)} \end{cases}$$

(No picture was drawn in the last case)

It is reported that the argument used by the students to decide the continuity of function $\boldsymbol{f_1}$ was primarily the fact that the function was given by a single formula. The argument used to decide the continuity of function $\boldsymbol{f_2}$, however, was associated to those "breaks presented in the graph", which they expressed by saying: the graph is not a single piece; the function is not defined in the origin; the function has an infinite jump in the origin. In the case of the function $\boldsymbol{f_3}$, many of the students responded: everything is a single piece!

The work developed by Sierra, González and López (2000), shows that 45.6% of a total of 145 students, lack a clear understanding of the concept of continuous function at a point. It is mentioned that the conceptions that the students have are linked to those that have appeared throughout the history of mathematics and that some problems in understanding are a consequence of the teaching, among these, the excessive recourse to intuitive ideas and the use of algorithmic techniques to decide the continuity or not of a function. Is a work start by considering such problems associated to the teaching and learning of the concept and by trying to give the concept an epistemologically different treatment to that given in textbooks and the aforementioned research. One of the authors' questions from the study is given here: Which of the following graphs (figure 2 and figure 3), represents a continuous function on the interval [a, b]? Justify your answer.

Figure 2

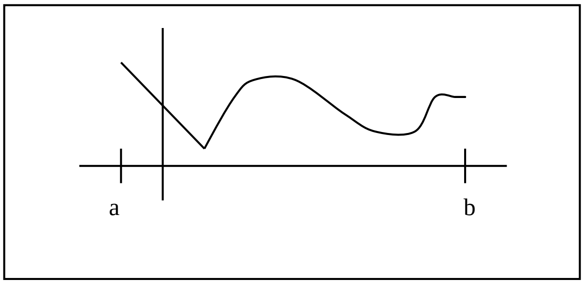
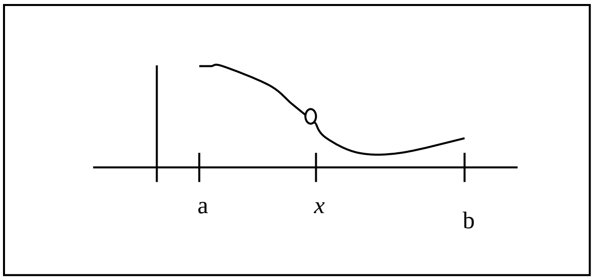


Figure 3



In this sense, starting from a brief historical and epistemological review, we examine the way in which the concept function and continuous function at a point have been developed.

In this respect, is show how Arbogast distinguished two ways in which the continuity of a function could get lost, as reported by Edwards, (1979):

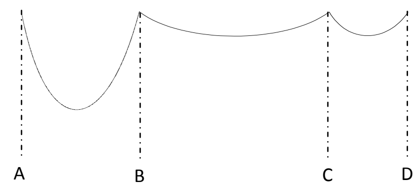


Figure 4. The graph changes form

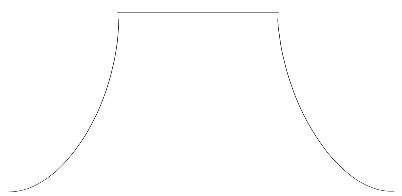


Figure 5. Parts not joined in a curve

1. He considered a function in such a way that the interval [A, B] was given by a portion of a parabola, in [B, C] for the portion of an ellipse and in [C, D] for the portion of a circumference (Figure 4).
2. He argued that such a function did not obey a law of continuity, that is to say, a permanency of form. Therefore, this curve was conceived as discontinued.

A discontinuous drawing of a curve was preconceived as the second way that continuity could be lost. The functions that expressed this type of curve would be understood then as discontinuous (Figure 5).

This way, is foresee that the perception of global continuity and the uses of punctual discontinuity precede the formal definition of punctual continuity. The concept of such continuity, as it is known nowadays, was developed systematically around the end of the eighteenth, early nineteenth century in central Europe with the works of outstanding thinkers like Arbogast (1759 – 1803), Euler (1707 – 1783), Bolzano

(1781 – 1848), Cauchy (1789 – 1857) and Weirstrass (1815 – 1897). It is believed that Euler (1707 – 1783), expressed certain sui generic conceptions regarding the concepts of function and continuity. He said: "A function of a variable quantity is an analytic expression composed in any way of this variable quantity and numbers or constant quantities".

Visibly, toward the end of the 18th century, the conception held of continuity, was related to the perception of space and the notion of the graph of the function, that is to say, the continuity was conceived as a permanence of form. An object is continuous if it does not manifest any sort of interruption. In this sense, a curve would be characterized by means of the connections or continuities of its trajectory. Thus, we say that punctual continuity arises as the result of the pathologies of global continuity. The notion of punctual continuity or continuous functions at points has been developed in strict relation to the way the concepts of function and continuity are conceived. For example, two interpretations of these concepts have been identified: the first is associated with the intuitive notion of correspondence of values; and the second with an algebraic expression or formula (Farfán, 2012; Ferraro, 2000). However, these notions do not become predominant until it is possible to conceive an algebraic expression or formula that relates the algebraic with the geometric. In this context of expression, a problem arises: the study of the notion of continuous functions in reference to a point. Is suppose, therefore, that by considering epistemological, cognitive, didactic and socio-cultural aspects in a systemic way, just as suggested by the theoretical socioepistemological approach, it is possible to redesign mathematics school discourse on the concept of punctual continuity.

Toward a didactic approach

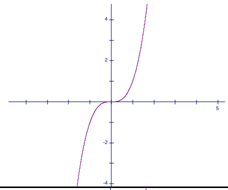
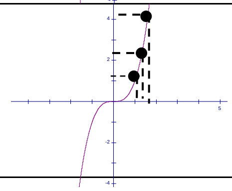
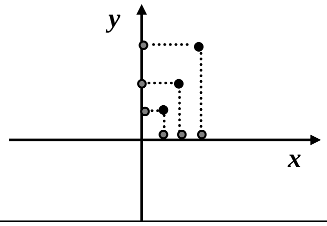
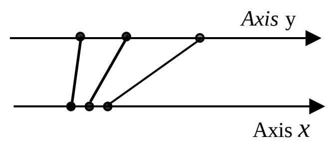
When researching how the concept of punctual continuity is introduced and taught in the classroom, we find that when punctual continuity is introduced in calculus course, it is neither taught based on the explanation of the notion of punctual discontinuity nor in the global perception of continuity, which, as we say, occurs naturally in the spontaneous reasoning of students. This type of school treatment creates difficulties for the learners. Our thesis then consists of accepting that the notion of punctual discontinuity and the global perception of global continuity should precede the treatment of punctual continuity. To do that, it is necessary for thought and the language of variation to be developed among the students with regard to the notion of continuous function at a point, as well as the handling of diverse forms of representation.

It is describing here an experimental pedagogical design which offers a set of elements that can be considered in the generation of educational proposals. Such a design supposes that the understanding of the notion of punctual continuity is associated with a global understanding of the idea of discontinuity that conflicts with punctual continuity. This suggests that the notion of punctual continuity is only grasped by the students when it appears as a way to avoid or study discontinuities of a punctual nature. The implementation of the design was carried out with eight university students between 19 and 21 years of age (five women and four men). The basic requirement was to have certain knowledge of the elementary functions of calculus as well as previous contact with the mathematical concept of continuity. Pencil and paper, blackboard and Sketchpad 4.0 software were used to design a series of activities that were presented on the computer screen according to three previously designed phases: preparation for the reading of the activities, development of the sequence and institutionalization of knowledge.

The first phase was intended to develop the necessary skills among the students for the appropriate reading of the proposed activities. To do that the students were shown a sequence of projections that showed a well-known graph (the cubic function) and the consideration of three arbitrary points on it, as well as the respective projections on the coordinate axes. Finally, the same points were shown over parallel axes, one above the other, which we will discuss in more detail later. The second phase aimed to create a specific scenario where the students discussed the notion of punctual continuity starting from the global perception of continuity and the idea of punctual discontinuity, by means explaining functional expressions associated with the dynamic representations seen on the computer screen; here, they formed teams of four. The body of the pedagogical sequence was formed by three activities on the computer screen.

Development of the pedagogical design

The considerations for the introduction the reading of the activities on the computer screen:

1. We begin with the presentation in transparencies of the cubic graph $f(x) = x^3$ in the habitual way, that is to say, on the coordinate axis (Figure 6).	Fig 6 
2. Three points were considered on the graph (sequentially, for better visual appreciation). We immediately went on to the representation of these points in their respective components that is to say, expressed by means of their ordinates and abscissas in the coordinate axes (Figure 7).	Fig 7 
3. The graph was then removed and only the points were left, in order to show more clearly the relation between the points of the graph and their respective "projections" – values in the coordinate axes – (Figure 8).	Fig. 8 
4. Finally, the location of the points was shown by means of their projections, so that the position of the points was determined by the location of their projections on the respective axes, which were placed parallelly (*y* axis over *x* axis), similar to how they would be displayed on the computer screen (figure 9).	Fig. 9 

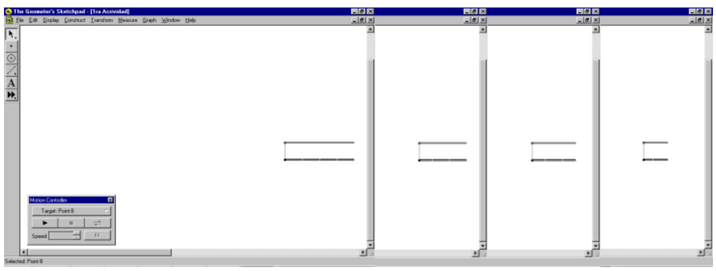
After this activity, the students were able to interpret the representations that would be shown to them on the computer screen. Next are mention the codes used in the transcriptions and some segments of learning episodes, with the purpose of recreating and identifying the discursive forms used by the students around the notions of discontinuity and punctual continuity.

- The letters A_S at the beginning of each intervention refer to the student who participates in the dialogue. For example, A_S alludes to Susana.
- The use of the diagonal / in a statement or dialogue indicates very short interruptions of the students, approximately five seconds. The use of the double diagonal // indicates interruptions over five seconds.
- The use of curly brackets at the beginning of each dialogue indicates the simultaneous speech of the participants
  $\left\{\begin{matrix} AS \\ AR \end{matrix}\right.$

Our comments on a dialogue are given as a comment from the observer: and our observation is given briefly below.

Activity 1. Is there a real function of real variable associated with what you observe on the computer screen? Let us remember that the students were shown an animation, movement of points and their trajectory (gray line, *y* axis and a black line, *x* axis) on the monitor.

To read from right to left



How it is said and gestured

Team 1

AR: Point *x* is equal to point *y*, then

Indicating the movement of the points (black and grey) going over the line that unites them, indicating the relation between *x* and *y*

$$\begin{cases} \boldsymbol{AS}: \boldsymbol{yes}, \boldsymbol{yes} \\ \boldsymbol{AR}: \boldsymbol{yes} \end{cases}$$

AS: or point *x* is equal to point *y*, then the graph will be,

AR: They move at the same speed. The same black point projects in the same grey point. The function is

Looks intensively at the representation on the screen and simulates the closed movement with hands and fists.

AB: They form the same parallel lines.

Indicates with the hand how both lines are formed simultaneously.

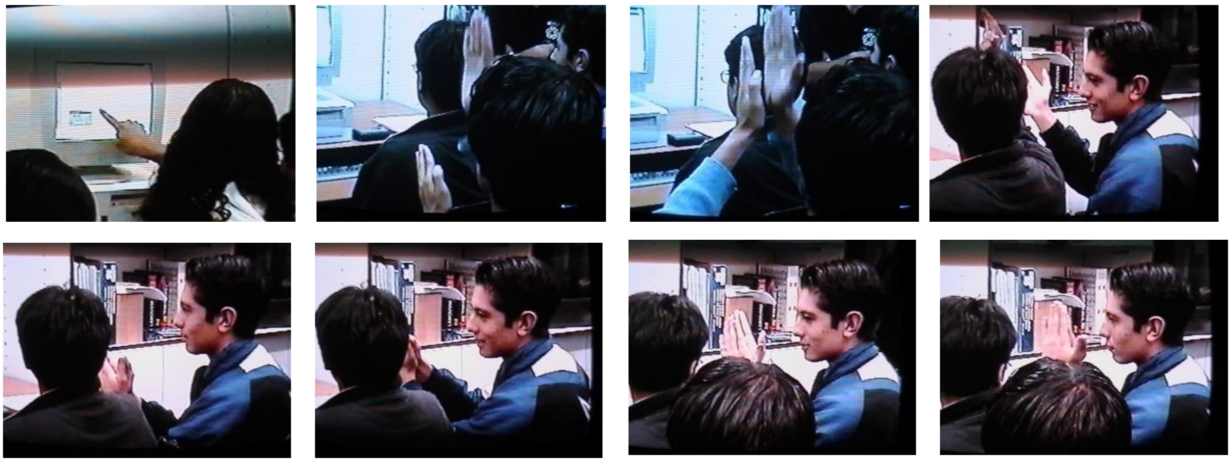
AS: It could also be the function's absolute value.

He says this at the same time as he observes the representation on the screen and makes slight movements with the hands, with very expressive eyes.

AR: Well that, too, but there it only goes from the starting point to there, that's why / there should be a frame of reference to start from, but as it is, it could be both of them.

AP: As he says, if we had a line in the middle that was representing a little more movement than the black point does, that would be *x* and we see then that this, the grey one, moves to the other side, then we would be sure that it is the absolute value of the function of *x*.

Pictures



Team 2

AO: It touches at the same time *x* and *y*… let's say that in the same value $y = x$, because if not it would move there and here but it touches it as well.

The student lifts her right hand as if simulating a knife (open hand, fingers together) and moves it from left to right slightly tilted, indicating movement)

AL: In the first one they are exactly at the same height, if we take one point and raise the perpendicular, then we find point y. The function found is $y = x$, is $f(x) = x$.

This student Al shows with his right hand the beginning of one point and then raises this imaginary point perpendicularly; finally, he moves the hand up slowly to the left, without touching the screen

AF: Let's say, that in the same proportion, *x* moves the same as *y*, that's why they are perpendicular.

Student AF expresses proportion with the index finger and the thumb up in the air. Then moves them to the right slowly and begins to express the perpendicular condition, moving the fingers together up and down. Student AL draws some kind of points with the index finger and the thumb together imitating the screen then moves slowly from one side to the other.

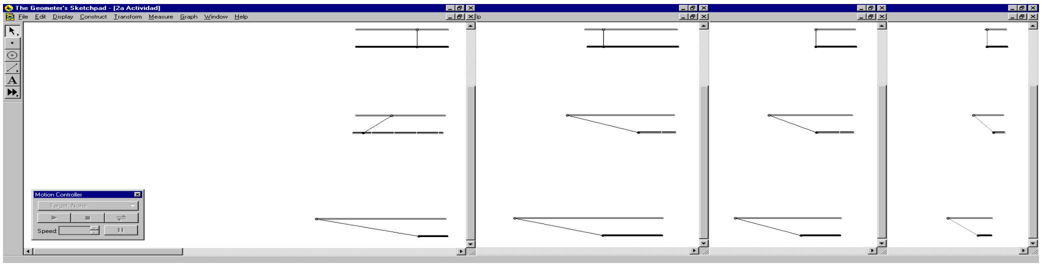
AO: Yes, because if a value were different we would see the variation. Yes, yes…

Smoothly shows different inclinations at different angles using the palm of the right hand

We could observe that the students did use some body language resources (particularly gestures), which allowed them to establish relationships between their mathematical knowledge and what they observed on the computer screen. This gestural aspect, enabled the students to not only talk about a function through gesture, but also, played an important role in the interaction among them, since it connected the types of representation – iconic, verbal and gestural – of a real function of real variable. We also observed that the use of different forms of discourse restores –in the case we are dealing with– diverse ways of using language in the communication of «mathematical» messages. Therefore, the specific forms of the discourse –description, presentation, narration and argumentation– necessarily, imply that people know and share the same «mathematical» culture.

Activity 2. In the three representations that you see on the screen, one is associated to function $f(x) = x^2$ in a certain interval. Which one? And why?

Reading right to left



Team 1

AS: It must be the one in the middle. Because we see in an interval (signals the third representation) when it reaches this point it's as if the y variable stops, while variable x continues its movement. After that they both continue their movement. So it couldn't be the square function f(x) = x^2 because in two equal values they should never be the same.

AS Signals the left side of the black line (x axis)

Intervention of the observer:

If you consider that the function is defined -2 to 2; what do you think?

AP: // If it arrives here, say, it is 2; if we applied the one of x^2, then it is 4

AR: // so it is the second representation because in both extremes it must be the same point.

AB: // Well, if this were a zero (alludes to the second representation), presumably the one above would have to be zero … It would have to be a straight line, wouldn't it? And if you look closely there is a moment there… well it is too fast, where it is almost a straight line.

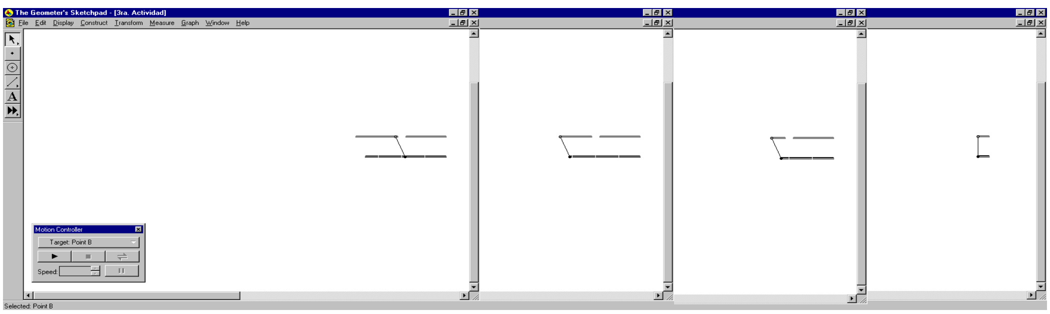
AR: // The right end and the left one must be the same point for this one; in the middle which must be zero, it gives zero, taking into account that this one is zero and this one, four.

Here the students AP and AS signal the black right end, which corresponds to the grey right (of the second representation), and the black left end corresponds to the grey right.

In the second activity, the students had to look for and establish reference marks on the representations on the screen, so that what was presented could be translated to a well-known mathematical language and managed by them. For example, they began to analyze the speed with which both points move, representing this fact with the movement of their hands. Operatively, when working in a direct way with their hands on the computer screen, they determined key areas in these representations, with which they enlarged and transformed the extracted information. Thus, the representation of a function ceased to be a mere graph, algebraic expression or table of values, because it became an iconic and gestural state. These last ones were the facilitators and intermediaries in the understanding of the mathematical knowledge.

Activity 3. Will there be a real function of real variable which describes to us what is seen on the screen?

Reading from right to left



Team 1

As: A function that describes the movement of the grey line and the black line!

The student As thinks carefully, she takes her right hand to her face.

AR: Then $f(x) = x$ until $x = 0$, supposing that the center is the zero (she refers to the center as halfway along the black line)

$$\begin{cases} \text{A}_S\text{: It is like continuity} \\ \text{A}_P\text{: Eerr…discontinuity} \end{cases}$$

Their faces show a certain degree of certainty in what they say, but something seems to worry them.

As: But there is a discontinuity there, isn' t there? It is supposed that here y doesn't have a value for x.

She points out the space where the grey line ends; her face indicates to the others that she already has an answer to the question. However, she also shows that she perceives something that she is not able to explain yet.

AR: No! It has two values for x, as the function of absolute value, that is to say, it intersects this same one.

The student AR looks attentively at the representation on the screen, trying to show a correspondence in the gap although it is not seen.

AS: In the origin it is cut.

His face shows that he is not only watching the screen, but he is also making comparisons: that is noticed in the way he observes. AP points out half of the black line and makes it correspond with the gap in a perpendicular way. He withdraws a moment and repeats the previous gesture again, only now it is noticed that his notions or thoughts are getting deeper, while his other companions continue debating.

AR: You have to find the first grey line and then the second. The first one is $f(x) = x$ until point $x = 0$. Now from left to zero something happens that transforms the function.

He indicates half of the black line and directs his finger perpendicularly to the part where there is no grey line. At the same time, he draws up on the screen a coordinate axis that has the cutting point (origin) in the black line underneath the gap in the grey.

AS: It is discontinuous in zero.

AR: Right, so, which one is it? It moves at the same speed, so there is no variation of anyone which moves faster.

AB: But there it is skipping the origin, isn't it?

AR: // Then it is going to be added or subtracted to a constant … (she said) …for $x < 0$, $f(x) = x - 1$ and that's it, isn't it?

At this moment this student in order to make sure of which he says, starts working on the work sheets, establishing the following explicit function.

$$f(x) = \begin{cases} x - 1, x < 0 \\ x, x > 0 \end{cases}$$

The less than equal is not determined for it to be a function. If it belonged to both of them it would not be a function // it is not defined in zero [meaning that only one of the parts of the function takes the value zero, but she doesn't know which one and, therefore, the ≤ sign will not be used . They say it is not defined or remains undefined but the function does take the value zero, that is, it is defined for this value.]

Team 2

AO: Look how it goes straight, and here it goes…(indicates the tilt of the line that joins the points of the black line with the grey line)

With the index finger on the computer screen, she traces the grey line up to the "breaking point" and when it passes by the place where the line breaks, immediately tilts the hand slowly to continue with the trace.

AL: It had to be a function defined in two parts.

The student observes the behavior of the grey line and infers the answer, his face shows great confidence

AO: Yes, isn't it? It cannot be one alone. I mean!

AO: // Here $y = x$ (signals the first part of the right to left of the grey line)

A_F: But, look carefully!

A_L: It varies in the same proportion (for $x > 0$)

AL: // If it is constant, you add up (in sections)

It would be $f(x) = \begin{cases} x - a, x < 0 \\ x, x > 0 \end{cases}$ and we do not know whether it is open or closed

Pictures







In this activity, the students combine everyday verbal expressions with their own phrases of the mathematical language, in order to talk about the notion of the non-continuity of a function in some specific value of its domain. Phrases like "it breaks", "it jumps" and "it bounces", tied to gestures, hand motions, the smooth and not smooth displacements of the hand from one side to another, as well as the perpendicular indications with gestures, are intended to be a means of connection between a mathematical knowledge and a particular form of communication. The students’ expressions provided a sample of the relationships established between the mathematical content (punctual continuity) and the discursive aspect.

Notice that, until this phase of the development of the activity, it has not been necessary to appeal to the accustomed “school scenario” to discuss the ways a mathematical knowledge is built, or the notions and conceptions that the students have of a certain concept (the continuity of a function). In other words, we suppose that the description of scenarios where they venture gestures, visual and discursive aspects around a notion, favors the emergence of elements of decisive analysis that enables them to expand and recognize certain processes linked to the understanding of a notion. For example, we infer that the visualization process can be characterized by means of the gestures used in a given problematic situation.

Conclusions

The results of our pedagogical design, provide significant information on the contribution of gestures in the re-definition of the concept of continuous function at a point. We identify some elements that were essential in the formulation of the students’ responses, at individual as well as group level; for example, the incidence of the gesture aspect is a means that offers the possibility to articulate the visualization of mathematical concepts, so that a representation on the computer screen only establishes the scenario where the student will expand and generate new meanings even more, if these representations are articulated with gestures.

In this sense, our didactic design offers an alternative of analysis in the study of the notion of punctual continuity where the understanding of such a notion does not come from its group understanding, but from the idea of discontinuity that creates a conflict with the notions of punctual continuity. We show that, in fact, when the students faced the notion of global continuity and punctual continuity, this allowed them, on one hand, to generate discursive mathematical arguments, and on the other, to establish the idea of punctual continuity. Among the first arguments, it is the use of the analogy, the resource of the metaphor and gestures as antecedents to the mathematical resources. Let us mention the cases, in which the students used such linguistic expressions as “it jumps”, “it bounces” and “it breaks” accompanying them with gestures, to finally associate them with school knowledge. Therefore, in order to allow students to build new meaning and construct mathematical notions, at the same time as new elements of analysis emerge to understand the way learning takes place, it is necessary to place the student in a scenario where he or she has the freedom to express him or herself using discursive expressions as well as gestures.

We have been able to identify that the students recognize a function in diverse representations, and from there develop the necessary competences to analyze the property of global continuity and punctual continuity. The epistemological and social information on the construction and development of the mathematical notion benefitted our design and analysis. Lastly, we will say that the analysis of the gesture in connection with the discourse promotes learning, starting from the established interactions between the members of a group and shared knowledge. Gestures as a part of the discourse offer elements for the study of the processes linked to the production of learning.

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