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An Aspect of Generalization Act from an Actor-Oriented Transfer Perspective*

Genelleme Eylemine Öğrenen-Odaklı Transfer Perspektifinden Bir Bakış

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Abstract

Generalization is one of the most important algebraic actions. The development of students' generalization skills is an important goal of mathematics curricula. Being aware of what happens in students' minds during a generalization action is thought to increase teaching quality. The actor-oriented transfer perspective explains how students relate new knowledge with existing knowledge. Researchers focus on what happens in students' minds during this process. Based on psychological models of knowledge and mental act of generalization process, a cognitive model on generalization from actor oriented transfer is provided in this study. According to the model, students relate similar situations and search whether the pattern is constant, then extend the rule beyond the given situation. These stages constitute the ways of thinking of the generalization act. The products obtained as a result of these processes are the ways of understanding of the generalization act. To this end, this paper aims to investigate several approaches to students' generalization processes and the relationship of the actor-oriented transfer with these approaches.

Keywords: Generalization, actor-oriented transfer, teaching experiment, cognitive model.

Öz

Genelleme, en önemli cebirsel eylemlerden biridir. Öğrencinin genelleme yeteneğini geliştirme, matematik öğretim programlarının önemli amaçlarındandır. Bu sebeple, genelleme yaparken öğrencinin zihninde olup bitenler hakkında fikir sahibi olunmasının yapılan öğretimin niteliğini artıracağı düşünülmektedir. Öğrenen odaklı transfer, öğrencinin karşılaştığı bilgiyle zihninde var olan bilgisi arasında nasıl benzerlik kurduğunu açıklamaktadır. Dolayısıyla öğrencinin zihninde gerçekleşenler, araştırmacının odağındadır. Bu çalışmada bilgiye ve genelleme eylemine ait psikolojik modeller temele alınarak, öğrenen-odaklı transfer yaklaşımı açısından genellemeyi açıklayan bilişsel bir model sunulmaktadır. Modele göre öğrenciler benzer durumları ilişkilendirerek örüntünün sabit olup olmadığının araştırmakta ve kuralı verilen durumun ötesine genişletmektedirler. Bu aşamalar genelleme sürecinin düşünme yollarını oluşturmaktadır. Zihinde gerçekleşen bu süreçler sonucunda elde edilen ürünler ise genelleme eylemine ait anlama yollarını oluşturmaktadır. Bu çalışmada, öğrencilerin genelleme süreçleri ile ilgili yaklaşımlar ve modelin bu yaklaşımlarla ilişkisi de bu makalede incelenmektedir.

Anahtar Kelimeler: Genelleme, öğrenen-odaklı transfer, öğretim deneyi, bilişsel model

1. Introduction

Most studies conducted in the field of mathematics investigate what happens in students' minds to better determine what mathematical thoughts are being carried out as they work through a particular problem. Consequently, those studies that propose a model of students' mental processes are of particular importance. Accordingly, this study proposes a model of the generalization processes employed by students when working through those algebraic problems presented

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herein. Like any students' mathematical-thinking model, one cannot claim that the model presented is generalizable to all students. Belief, motivation, language, and context are all influencing elements of students' cognitive generalization processes (Noss & Hoyles, 1996). However, the model introduced here is limited to what happens in students' minds. This model is thought to be useful in terms of serving as a general frame about what happens in students' minds during a generalization activity.

1.1. Generalization

All new concepts are formed by abstracting their unique characteristics in the processes of perceiving and comprehending these characteristics. Concept formation is carried out through the three-stage process of perception–comprehension–concept. The first stage of this process perception, refers to the individual's sensory experience of the external world and their ability to visually conceptualize something they have not yet encountered using their visual, auditory, and kinesthetic experiences. Generalization and abstraction are performed during the second 'comprehension' stage of this process. For example, when an individual hears the word "table" they will associate typical characteristics of tables—as well as objects of that kind or class—of which they have previous experience. The generalizations and abstractions performed during the comprehension stage are limited since similar characteristics of an object are focused as a group. These comprehensions developed by students are generalized in a way selecting general elements from a particular element of that comprehension. The characteristics determined are represented through particular terms. Therefore, the generalization made is produced in a precise and complete way; thus, the concept is formed. Generalized concepts are abstracted in this way and, as a result, become the products of the abstraction process (Davydov, 1990).

This process of generalization, which may manifest in several ways, constitutes the essence of mathematics (Mason, 1996). Existing literature in the field suggests a number of definitions for the generalization. If a definitive or comprehensive definition is to be extrapolated from these suggestions then a generalization is the process of determining of similarities between those conditions forming an argument, moving the argument beyond these aforementioned conditions, and then extending the argument to patterns, operations, structures, and relations among them (Ellis, 2007; Harel & Tall, 1991; Kaput, 2000; Polya, 1957). There exist several opinions as to the development of this process. According to Polya (1957), a generalization process occurs gradually and follows the stages that explain the situation observed, gives examples regarding the situation, and examines special examples. In addition, a generalization should be absolutely ended up with a mathematical proof. On the other hand, according to Radford (2003), generalizations occur in three stages. The first stage concerns factual generalization and indicates that the process of generalization takes place at a physical level. Throughout this stage, generalizations are usually carried out in an operational manner. The second stage, the contextual generalization stage, incorporates a more abstract language utilization in defining the generalization. Within the contextual generalization stage, students form various interpretations about the next term on the figures they have seen. The final stage, the symbolic generalization stage, involves students making algebraic demonstrations using the letters and expressing their generalizations.

Garcia-Cruz and Martinón (1998) investigated students' levels of generalization in three stages. During the procedural activity stage, students notice any repetitive or iterative characteristics of a pattern and can detect the common difference among the strategies used. This is the only generalization made at this level. The formation of a generalization indicates the assimilation and organization of the stimuli concerned; therefore, the knowledge gained by the student corresponds to the generalizations they have constructed. Sometimes, the assimilation is formed into an incorrect schema, which leads to an inaccurate generalization. At this stage, a particular rule is imposed so that operations can be undertaken and—if the rule is correct—procedural comprehension is then achieved. The next stage is the conceptual comprehension stage wherein students apply a similar action to a similar problem and develop a strategy as a product of generalization. The rule established in the previous problem now acts as a stimulus. This stimulus is included within an appropriate schema per the assimilation–organization processes and, as such, a strategy developed in regard to the students' performance throughout this process. This strategy is subsequently used in a new and similar problem; thus, the conceptual comprehension is achieved (Garcia-Cruz & Martinón, 1998). The formation of an idea from the perceptual to the conceptual; this developmental process from the concrete to the abstract is referred to as conceptual generalization. A generalization in which the relations and connections among objects, and mental analyses are made are at the scientific level that requires a higher level of thinking are referred to as theoretical generalizations (Davydov, 1990).

Generalization is one of the most important algebraic actions. According to Vygotsky, students who succeed in algebra display a higher level of thinking enabling them to make generalizations and abstractions (As cited in Schmittau, 2011). Consequently, an argument can be made that algebra should be taught to students throughout their early years (Radford, 1996). Teaching generalizations should assume the following procedure: relational and pattern analysis, systematic organization of knowledge obtained, and expressing the relation in a verbal and symbolic manner (Bell, 1995).

1.2. Classification of Generalizations

Stacey (1989) categorized generalizations into two groups: near generalizations and far generalizations. While the former is concerned with finding the following algebraic term, the latter is concerned with identifying the rules of patterns. Stacey (1989) identified three generalization strategies: the iterative strategy, in which the common difference is added to a term of a pattern in order to find the next term; the strategy of searching a functional relation, in which a mathematical expression is identified for the relation between the terms of the pattern; and, finally, the strategy of extending to the whole, in which rational reasoning is used in considering the ratio of $f(x)=nx$, wherein $f(x)=ax + b$ $(b=0)$.

Dörfler (1991) asserts that there are two kinds of generalization—experimental generalization and theoretical generalization. Experimental generalization involves the identification of similar characteristics among some objects or conditions and the description of these characteristics as the general properties of similar objects or conditions. Similar characteristics are obtained by comparing the shapes and appearances of objects, and numerous generalizations of objects are made in this way. For example, the triangle concept represents similar and general characteristics of triangular objects; an individual first compares the triangular object and non-triangular objects and forms the concept 'triangle' by distinguishing similarities among triangular objects such as—for example—the necessary property of having three sides. The characteristics of a triangle are then constructed in the mind by concreting them. In this way, the characteristics concreted are formed as the concept 'triangle' by abstracting triangular objects from other objects. Experimental generalizations are limited as they are dependent on the appearance of objects—hence, theoretical generalization is both more suitable for the purpose while also allowing for a greater degree of development. Within theoretical generalizations, a concept is abstracted based on the relationship among all elements of that concept, rather than it being generalized according to similar characteristics of concrete examples. Bills and Rowland (2009) similarly classifies the generalizations that are made according to the similarity of objects as experimental generalizations and generalizations that are made considering the similarities between objects and structural characteristics as structural generalizations.

Harel and Tall (1991) define generalization as "the application of an argument to a broader content" suggest three kinds of generalization: expansive generalization, indicating the extension of an existing schema's applicability without creating a new schema; reconstructive generalization, indicating the reconstruction of the existing schema to increase its applicability; and disjunctive generalization, indicating the formation of a new schema while transforming the schema to a new content. Although a disjunctive generalization is thought to be a successful generalization, it is nevertheless differentiated form other kinds of successful generalization as it does not contain a particular example of the general situation. Though it is unable to realize effective learning, expansive generalization is comparatively easier than reconstructive generalization (Zazkis & Liljedahl, 2002).

Generalizations are the essence of algebraic thinking. The *procept* concept that is an effective algebraic procept (Gray & Tall, 1994; p. 95), while also being representative of the process and product. It is also an indispensable tool in the symbolic representation of mathematical modeling, problem-solving, and quantitative relationships (Becker & Rivera, 2006). Generalization itself is considered to be a product or a process, though it is actually both; a generalization expression is obtained at the end of a generalization process (Yerushalmy, 1993). Considering generalization as a process and product is also consistent with the nature of mathematical knowledge which incorporates two dimensions: conceptual knowledge and operational knowledge. It is inevitable that all mathematical knowledge—cognized in the mind and based on an act of generalization action—will incorporate these two dimensions. According to Harel (1998), every mental action we perform is comprised of the ways of thinking and understanding. An individual's actions or expressions are performed as a result of his or her mental actions; these subsequently comprise the products those mental actions, which, in turn, comprise the ways of understanding of the mental action performed. These ways of understanding revealed similar cognitive characteristics to one another, and these characteristics are referred to as ''ways of thinking' of the mental action being carried out (Harel, 2008).

According to the current literature, students' actions such as finding a term of a pattern, for example, by counting, continuing a pattern to a close step, or using a functional relationship are called the strategies students use while making a generalization. The strategy in this context is, in fact, a cognitive characteristic of our mental actions. These strategies may be considered as ways of thinking about generalization as they constitute similar cognitive characteristics of the generalization action. Therefore, the ways of thinking of the generalization action are determined as iterative thinking, explicit thinking, visual thinking, numerical thinking, and pragmatic thinking, within the existing literature (Becker & Rivera, 2005; Lannin, 2004, 2005).

Recursive thinking is a technique used widely in pattern generalization problem-solving. Recursive thinking is how a series is investigated in terms of whether the common difference between a series' sequenced terms is valid for every term in the series or not (Lannin, 2004). Even if recursive thinking is the first method we use while finding the rule for a pattern, doing the same operation repeatedly is not sufficient. Therefore, one might argue that explicit thinking is more effective and efficient than recursive thinking. Explicit thinking involves calculating the value of dependent variable according to the given value of the independent variable. Generally, algebra lessons given in schools involve finding a formula for a given problem. However, students may not necessarily have the required mathematical knowledge to find the correct formula. Therefore, while finding a rule the explicit and recursive thinking proceed as nested (Lannin, 2004, 2005).

Visual thinking is defined as *explaining the shapes in a way to complete the series, even if they are not visible.* Students who think visually focus on the structural properties of a shape (Becker and Rivera, 2005). For questions that require one to consider the structural characteristics of a shape, the focus can quickly move towards a "how many" question. Becker and Rivera (2006) state that those who prefer a quantitative approach while making generalizations use quantitative operations to find a rule. This approach requires turning the shape pattern of data into quantitative patterns and using these quantitative patterns to find a rule (Becker and Rivera, 2005; Tanışlı and Yavuzsoy Köse, 2011). The use of visual and quantitative thinking methods together is called pragmatic thinking (Kirwan, 2015, p. 29). Therefore, while finding the rule of a given pattern, applying a pragmatic approach is beneficial.

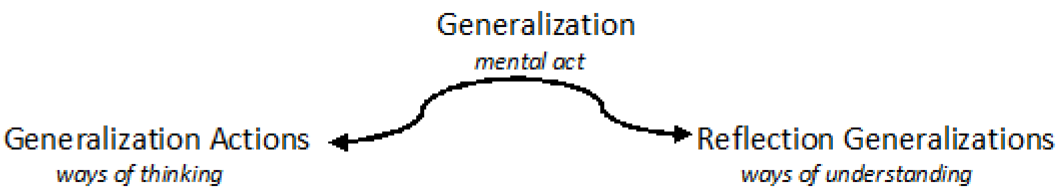

Figure 1. *Components of Generalization Act*

Ellis (2004) regards generalization as a process (generalization actions) and a product (reflection generalizations), and has formulated a taxonomic categorization of generalizations. Lobato's (2003) actor-oriented transfer perspective (AOT) was used as the basis for this taxonomic system. The concept of transfer remains an extant subject of research in both educational and psychological fields. The transfer perspective deals with how an individual constructs new knowledge, as well as the relationship between this new knowledge and existing knowledge. Utilization of the knowledge obtained in the following situation constitutes the scope of the studies conducted on transfer issues (Nokes, 2009). Lobato (2003) employed a learner's perspective when investigating the processes students use in constructing similarities among the problems they encountered, opening up an original dimension within transfer literature by shifting the perspective from researchers to learners. AOT provides significant clues on how individuals relate new situations to existing structures in their minds (Lobato, 2003). According to the literature, the processes of generalization and transfer are similar. An individual's extension of their reasoning and the process of developing a rule are the most prominent characteristics of generalization definitions that possess notable similarities to the development process of generalization and learning within AOT. Furthermore, the action of expressing similarities between conditions performed within a generalization process, and the action of investigating how an individual relates new situations to existing mental structures within AOT are similar (Ellis, 2007). However, the aforementioned generalization strategies can be mostly explained using the researcher-based transfer perspective because such strategies are unable to reveal any information on how students relate new problems to previous situations. For example, Stacey (1989) found that students used the counting strategy enabling them to find the next figure in a pattern question and expressed this strategy as "the number of lights in each figure increased by four". The counting strategy does not, however, reveal any information on what previous knowledge these students used and related with the problem. Now information has been provided on AOT, the following sections shall outline a taxonomy based on AOT, generalization actions as a process, and reflection generalizations as products.

1.3. Generalization Actions as Ways of Thinking

What happens in students' minds while making a generalization is characteristic of the cognitive process taking place. Ellis (2004) defines this process as the characteristic of an individual's process of generalization in his or her mind during his or her verbal or written action and classifies these as "generalization actions". For example, a student can solve a pattern problem by "relating" it to a previously encountered pattern or else by "searching the similar pattern". These are the processes taking place in the student's mind and generalization expression is revealed at the end of these processes. Ellis (2004) categorized generalization actions under three different headings: relating, searching, and extending. The boundaries among these categories are vague and they all constitute part of the generalization process. Students' focus is considered in the formation of these categories.

The actions performed within the 'relating' category are random, rather than being intentional. The relation is constructed between two or more problems, situations, or mathematical objects. In addition to these relations, Oflaz (2017) reported that students make relations with more knowledgeable external sources such as teachers seen as an authority and textbooks. As students search for a relation, they are not aware of how such a relation was constructed.

Students sometimes search for similarities among various examples with an awareness of a similarity relationship. When students repeat operations again and again to find a similarity, this process falls within the ‘searching’ category. A similarity relationship being searched concerns the similarity of the pattern, procedure, or solution in question. Oflaz (2017) stated that students might perform the action of searching the same piece investigating whether the terms composing the pattern are constant. The actions performed in this regard are more intentional. A student is said to performing an extension if the relationship is expressed in a more general form that is beyond the given conditions while the student is also aware of a similarity relationship. Products such as a new relation, structure, or definition are yielded as a result of the extension.

1.4. Reflection Generalizations as Ways of Understanding

The ways of understanding of generalization actions refer to the product which is the generalization expressions revealed as a result of the generalization process. Students’ expressions created as a result of the generalization process are classified as reflection generalizations. Reflection generalizations consist of verbal or written expressions put forward by students at the end of generalization actions. Students can produce a pattern, rule or definition at the end of the generalization process.

Students can make a generalization in the forms of a pattern, property, rule, or strategy. The generalization made is identified through a definition or a mathematical expression. Students can express their generalizations as general principles such as general rules, patterns, or strategies. Expressions such as these are the expected products of students since they are also in accordance with algebraic representations. These generalizations made are also valid for mathematics researchers.

In some situations, students formulate sentences by indicating that the expressions constructed as a result of their generalizations comprise the basic characteristic of the pattern, relation, and class. The scope of the phenomena defined can, therefore, be extended. The student who extended a property or relation is then able to define a class of conditions in which this property is applied. This class might not necessarily reflect the entire class mathematically; nevertheless, it is sufficient as a class to which the property is applied from the students’ perspective. The expressions describing the properties of an object belonging to a particular class identify the definition of that class. In some situations, students can apply a previous generalization to a new problem. Thereby, they not only make a generalization but also use it at the same time.

It seems important to know what happens in students’ minds when they make generalizations. With this way, teachers and researchers can help students construct their own generalizations. To this end, the aim of this study is to determine what happens in students’ minds when generalizing. Hence, generalization act was dealt with AOT.

2. Methodology

The teaching experiment methodology (Cobb & Gravemeijer, 2008; Steffe & Thompson, 2000) was employed in this study. Teaching experiments are conceptual tools that the researcher designs and organizes the teaching practices. In teaching experiments, how the student creates mathematical knowledge and the development of knowledge in the process are investigated. They are consisted of a series of teaching of practices (Steffe & Thompson, 2000; Steffe & Ulrich, 2013).

2.1. Participants

The study was conducted with 9 eighth-grade students. The researcher developed operational and conceptual algebra tests and applied them to a total of 167 eight-graders from the participating school. The students were selected in accordance with high and medium scores they received from algebra tests and the ability to reason in mathematics problems and of expressing his or her thoughts clearly for the study. Therefore, the study was conducted with a total of 9 students (four males, five females). To maintain the confidentiality of the participants’ information, pseudonyms were used throughout the study.

2.2. Data collection process

This study was carried out in three stages. At the first stage, classroom observations were made by the researcher, the teaching experiment was designed, and participants were selected. At the second stage, the teaching process was carried out, individual student interviews were conducted, and continuous analysis was carried out. At the last stage, retrospective analysis was conducted.

First of all to organize and prepare experimental teaching, the context is defined as patterns and relationships, analysis of change and equations. The most important thing to consider in creating the tasks was to determine tasks that would confuse students' minds and motivate them to solve. In line with the aims of this study, the students were expected to find different ways of thinking about the rules of the patterns given as shape patterns and numerical patterns, to realize the relations among the patterns given as multi-representation patterns, to examine the changes between the variables and to impress them in multi-representations. Therefore, the teaching experiment included 37 tasks. The experimental teaching process lasted 7 weeks, with the first week being used for pre-implementation and the remaining 6 weeks for the main implementation. Among the implemented tasks, 12 focused on patterns and relationships, 15 on analysis of change and 10 on equations. The first and second task are examined within this article (Appendix 1).

2.3. Data Collection Tools

Data of this research included the camera recordings of each session of experimental teaching, camera recordings of group studies and students' discussions, individual interviews with two students after each session, individual interviews with all student participants, students' worksheets, students' logs, and the observation notes of the teacher and researcher.

2.4. Data Analysis

The camera recordings were transcribed and analyzed using continuous and retrospective analyses. Content analysis was used to analyze the students' worksheets, logs, and the researcher's and teacher's logs.

2.4.1. Continuous Analysis and Retrospective Analysis

The overall data analysis consisted two stages: continuous analysis and retrospective analysis. The continuous analysis was conducted through the researcher's and teacher's evaluation of each stage (Molina, Castro & Castro, 2007; Simon, 2000). The retrospective analysis was the last stage of designing the teaching experiment. It is the process of evaluating the data within a more comprehensive theoretical framework (Cobb, Jackson, & Dunlap, 2014, p. 20).

2.5. Validity and Reliability

Validity and reliability studies of teaching experiments are required in order to talk about the scientific nature of these researches. However, the validity and reliability of such research are performed in different ways than experimental studies. The processes of theory, design, practice and measurement are spread over time and support each other. Therefore, the validity of the research has been provided (Design Based Research Collective, 2003). It is not intended to generalize the findings of the researches. In the case of giving detailed explanation about how the process is carried out, it is provided for readers to portray the process and to deduce how they can get results when applying similar drafts (McKenney, Nieveen & van den Akker, 2006). To ensure the validity of the research, expert opinions were taken from 2 mathematics teachers' and 5 mathematics instructor, and the final version of the experimental teaching procedure was prepared having done the necessary revisions. Triangulation is a significant technique for providing the reliability and validity of the research (Miles& Huberman, 1994); thus, the data of this study were collected using various data collections tools and included recordings of the sessions from the experimental teaching, the interviews with two students, worksheets of the students, student diaries, researcher and teacher diaries.

3. Findings

With the tasks applied during the experimental procedure, the students are expected to find different ways of thinking about the rules of the shape and numerical patterns, to identify the relations among the patterns given as multi-representation patterns, to examine the changes between the varieties and to express them in multi- representations. What happens in student's mind has similar features while generalizing. However, only the first two of the tasks that are determined in relation with patterns and relations will be investigated in this article in order to get a complete and detailed understanding regarding this process.

On the 1st day of the teaching of the experimental procedure, shape patterns were given to the students and various questions were asked. It was observed that the students had no problem with drawing the fourth and fifth steps of the first three stages of the patterns given. However, the statements on how the tenth step could be were inadequate. The most efficient statement about the first task was Gül's statement: "There will be as much block as the step we are at, I mean there will 10 vertically and 1 on the left bottom corner". Except her, there were no other true statements regarding how the 10th step may be while they were at the first three task. Ali and Bartu has found the rules of the patterns of 10th step and how many squares there will be at the 10th step instead of giving a statement about 10th step at each first three steps. Considering the students' answers, it can be said that the possible reasons of inadequacy in stating the asked steps can be the inability to visualize the shape in their mind or inadequacy in expressing what they have in their mind.

When it comes to finding the rules of the patterns, all the students found rules by exchanging the shape patterns into numerical patterns. For instance, Melike expressed the rule she found about the first task, like all her friends did as in the following:

Melike: When changing these to numbers, it goes as 4,7,10,13 and so on. It goes 3 by 3. So the odd is 3n. We need to add 1 to find the first term. So, we get 3n+1 when we add 1 to 3n. The result is the same in others too.
Researcher (R): How could you get that?
Melike: It's how our teacher taught it.

All the other students found the pattern rule in this way and they explained it as "it's how our teacher taught it". The actor oriented transfer gives significant clues about which cognitive structures they have in mind with the new situation the learner is confronting and how the learner is correlating it (Lobato, 2003). Thus, it's important what the student is focusing on at this process. The students are forming relations among two or more problems they confronted before, the features of status or the formal features at first when generalizing. They create these relations with more knowledgeable authorities such as the book or teacher. The students here have found the pattern rules by relating with the authority. They, however, could not recognize what has been mentioned when asked whether there is "another way to find" the rule or not. After that, Oğuz, Bartu and Melike examined each line and column separately with the guidance of the researchers.

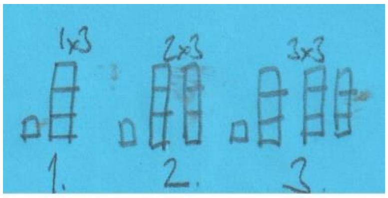

Figure 2. *Oğuz's worksheet*

Oğuz: It's increasing 3 by 3. The lines are staying the same but the columns are increasing. It's increasing according to the amount of term. If we name term number as n, 3 times n equals to it, not regarding the extra 1 square. I mean, for instance at first term it's 3x1=3, plus 1 gives the term number.
Oğuz has defined the constant and regularly increasing squares at each term. And then he has showed this relation as a rule by inferring the term number and square amount in term. This means Oğuz has found the pattern rule by determining the similar situations at each situation.
Sezen: I've realized this; there are 1,2,3,4, boxes here (the square amounts on the top lines of steps). There are 1,2,3,4 boxes here (the square amounts on the middle lines of the steps). And there are 2,3,4,5 boxes here (the square amounts on the bottom lines of the steps).
R: Can you form a rule out of this?
Gül: (The square amounts on the top, middle and bottom lines of the first step) this goes as 1,1,2; this goes as 2,2,3. So, we can say n,n,n+1. When we sum up, it makes 3n+1.

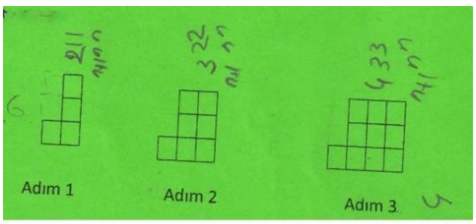

Figure 3. *Gül's worksheet*

Sezen and Gül has put the pattern into parts, has related the number of box and the step of the parts and formed a rule out of their sum. The act that the students had performed here is different than searching the relation among the terms that form a pattern. The students' concentration is on the parts that form the pattern. And what they do is determining whether the parts repeat at each terms or not and whether this changes are stable or not.

The students have tried to find another way to infer a rule other than the way their teacher taught on the other task given (Task 2).

Bartu: Let's add a square to all of them and make them a closed shape. For instance, if we another square into the center of the first shape, it will become a rectangle. And it will be a square this time, if we add 2 squares in the middle of the second shape. And the third shape will turn into a rectangle again if we add 3 more squares.
Ezgi: Like the rectangular numbers. Then it goes as rectangle, square, and rectangle.
R: What do you mean by rectangular numbers?
Ezgi: Remember we show it? There were numbers we write like the shape of a rectangle.
Ezgi has related numbers and shapes. It can be said that, Ezgi has connected back since she related a knowledge she learned before. Even though she has mentioned correlation in a wrong way, it's important what the student is focusing on. Examining the situation that is correlated is not obstructed even though what the student has correlated is mathematically wrong.
R: You got rectangles by adding little squares to each shape. How can you find the pattern rule?
Gül: We can find the total square amount by finding how many little squares there is and subtract from the squares we added.
R: How can we find how many little squares there are?
…
Ali: The edges of the squares in the rectangle are 1 unit. Then long edge of the first shape is 3 units and the short edge is 2 units. The others are the same.

Gül aimed to find the square amount in the terms by finding the square amount in the whole shapes that are formed by adding squares, and then by subtracting the added squares. By this way she has made a correlation among square amounts that are in the rectangle area. Ali, on the other hand, regarded one edge as 1 unit of the squares and stated that he can find the number of squares in this way, just as his teacher did once when solving a question. This means, Ali has stated that the area of the rectangle is equal to the unit square amount. Ali has examined whether this pattern works for all other steps instead of examining the rule he stated.
Melike: Let's find the areas of the rectangles. The incomplete area is equal to what we added, I mean there are 5 squares here, it's 1 more if we complete it; there's 7 squares there, 2 more than its not completed status. Then, if we don't complete it, it'll 2 squares less… 2 squares less the area.
Sezen: We completed the squares, this one's area is 6, we added 1 square to the first one, we subtract it back and it makes 5. We use the logic for the others, we subtract at all the others.
Gül: How can we find how many squares are there in one step? If one edge of a square is 1 unit, we can find it by multiplying the short edge and long edge. The short edge is 3 units and the long edge is 1 more than the step of the long edge here.
Melike: So the area is: short edge 3 units, long edge n+1 units. Then the edge is 3x (n+1).
Gül: But we added square to each shape. 1 square to first step, 2 squares to the second step, 3 squares to the third step. So n much to each step. If we abstract that it'll be: 3(n41)-1.
Melike and Sezen stated that subtracting the added squares from the rectangles is valid for each step. Therefore, they stated these operations are valid for near step and far step. Gül and Melike have stated their rule algebraically. By this way, they have completed their generalizing.
The students who found the pattern rule, then debated on other representations. Ali has stated the '3n+1' pattern rule in different ways like $\frac{9n+3}{3}$, $\frac{12n+4}{4}$, $\frac{15n+5}{5}$, $\frac{18n+6}{6}$. Oğuz has stated that there can be infinite rules found this way since multiplying or dividing of '3n+1' does not change the result. He has written the $\frac{12n+4}{4}$, $\frac{33n+11}{11}$, $\frac{3000n+1000}{1000}$ statements by mutliplying and dividing '3n+1'. This means, Oğuz has improved by going further on and expressed the rules in different ways.

4. Conclusion

Since the learner is at the center of a teaching and learning environment, studies should investigate what happens in students' minds during learning process. This is also the case for the studies investigating students' generalization processes. AOT explains how students relate new knowledge with their previous knowledge. Therefore, researchers focus on what happens in students' minds; nevertheless, analyzing the learning process that takes place in students' minds remains a significant challenge (Lobato & Siebert, 2002). However, actions performed in the teaching experiments in contemporary studies in the field of mathematics—including relevant literature in Turkey—provides important clues regarding what happens in students' minds (Lobato, 2003).

4.1. A New Generalization Model

Oflaz (2017) investigated generalization processes by developing a teaching experiment and applying it to eighth-grade students. The aim of that study was to determine ways of understanding and thinking within students' mental act of generalization. Considering the students' generalization of the algebraic problems presented within this study, generalization can be defined as "searching whether the pattern/rule is constant by relating similar situations and reaching an algebraic expression by continuing the rule" (Oflaz, 2017). This definition regards generalization in the dimension of product and process. Relating similar situations, searching to determine whether a pattern is constant, and extending the rule beyond the situation comprised the ways of thinking of the generalization actions. The algebraic expressions reached as a result of this process themselves constitute the ways of understanding of the generalization action.

In the literature, there are studies explaining the generalization process (Garcia-Cruz & Martinón, 1998; Polya, 1957; Radford, 2003). These studies do not provide any information about what happens in students' minds because they regarded generalization from a researchers' point of view. However, students can be guided toward a more efficacious development of the generalization process if researchers or teachers become knowledgeable about what happens in students' minds. As stated at the beginning of this paper, this study focusses on what happens in students' minds. Therefore, this study introduces a model of students' generalization process based on AOT.

According to existing studies in the field, generalization is regarded as a process or product. However, the generalization process as a whole is comprised of both what happens in students' minds while making a generalization (process), as well as the generalization expression that is revealed (product). Consequently, there is a need for a model investigating generalizations considering the process dimension and the product dimension together. This perspective also mirrors and parallels the perspective that our mental actions comprise of the ways of understanding and thinking (Harel, 2008).

In teaching experiments, models of students' mathematical knowledge can be presented based on students' actions and operations. Such a model is thought to be a guide for teachers since it provides information about students' mathematical knowledge (Cobb & Steffe, 1983). The students' generalization processes were analyzed in the teaching experiment implemented. Based on these analyses, the following model of the students' generalization processes can be presented.

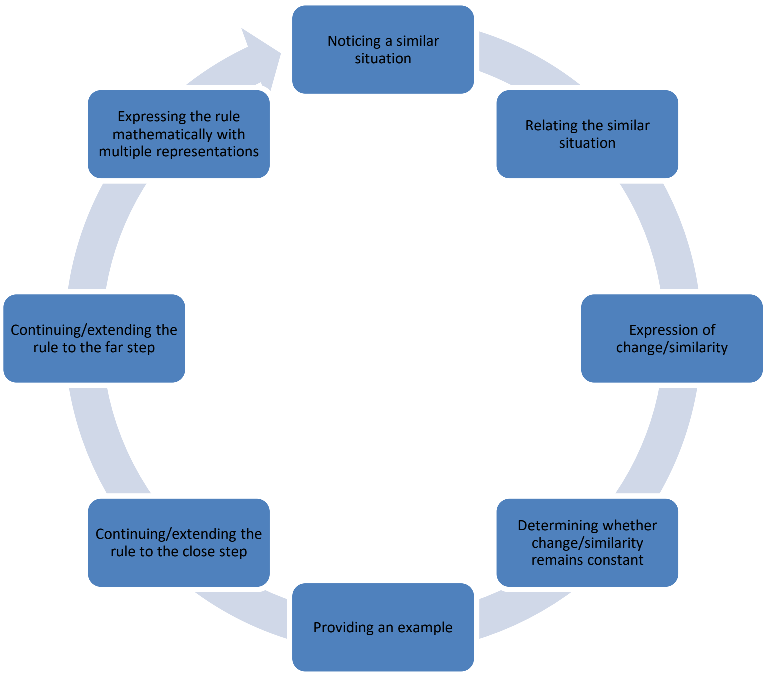

Figure 4. *Generalization Process in Accordance With AOT*

As can be seen in the model, students' generalization processes can be summarized as in the following: first, students notice a similarity by analyzing the situation, which is the first step of the generalization process. Students then relate this similarity to a problem situation, mathematical object, or authority in their minds. Students then commence the process of searching to determine whether this change is constant, and this process begins with the students' expression of this similarity. Students sometimes defend their ideas by providing examples while searching to determine whether this change is constant. The rule is deemed to be valid beyond the situation if the extension of the change/similarity is determined as constant to the close or distant step. The rule is then expressed using representations such as algebraic graphics or tables.

This study investigated various perspectives on how students undertake and make generalizations. Since the focus is on students and their processes of constructing knowledge, what happens in students' minds should also be understood while investigating their generalization processes. AOT explains how students relate new knowledge with their previous knowledge. Models of students' mathematical thinking do not have the characteristics of reflecting the truth completely or being correct for all students; thus, the model proposed by Oflaz (2017) cannot be valid for all students. However, the model is nevertheless thought to be useful in terms of providing a framework for what happens in students' minds as they make and form generalizations.

The steps of the aforementioned model are not consecutive; that is to say, a student may skip a stage and pass to the next depending on the content of the topic in question. This model provides information about what happens in students' minds while making a generalization. With this aspect, the model is thought to provide an idea for teachers regarding questions that are likely to be asked and the organization of the learning environment. The model might also be further developed through future generalization research within the related field.

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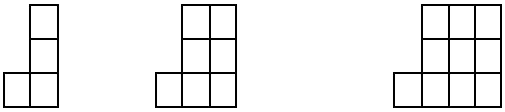
Appendix 1: Examples of the tasks

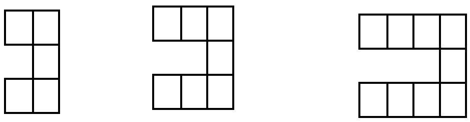
Task 1: Find the rule



Step 1 Step 2 Step 3

1. Please draw step 4 and 5 according to the pattern given above?
2. Please express what the 10^{th} step is like?
3. How do you express the rule of this pattern with one or two sentences? Please write this statement mathematically?
4. Please try to find the rule of this pattern in a different way?

Task 2 Find the rule



Step 1 Step 2 Step 3

1. Please draw step 4 and 5 according to the pattern given above?
2. Please express what the 10^{th} step is like?
3. How do you express the rule of this pattern with one or two sentences? Please write this statement mathematically?
4. Please try to find the rule of this pattern in a different way?